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are the equations which determine c, d and g; the solution of which can be performed by equations of the second degree, by first transferring the origin of coordinates to the centre of gravity of the n points.

My objection to Prof. Scheffer's solution is, that it does not recognize the proposition in relation to normals above referred to, nor the necessary and fundamental definition of the error of a point with respect to a line or surface, which definition is, that the error is the normal from the point to the line or surface.

SOLUTIONS OF PROBLEMS IN NUMBER FOUR.

SOLUTIONS of problems in No. 4 have been received as follows:

From Prof. W. W. Beman, 215; Marcus Baker, 213, 215, 217; George C. Comstock, 215; Geo. M. Day, 212, 213, 214, 215, 216, 217, 219; Prof. A. B. Evans, 213; Geo. Eastwood, 218; Henry Gunder, 212, 213, 214, 217; W. E. Heal, 212, 213, 214; Wm. Hoover, 212, 213, 214; Henry Heaton, 212, 213, 214, 215, 216, 217, 219, 220; George H. Harvill, 213; Prof W. W. Johson, 217, 220; Prof. J. H. Kershner, 212, 213, 214, 215, 216, 217, 219; Prof. D. J. Mc Adam, 212, 213, 214, 217; L. W. Meech, 218; Artemas Martin, 220; Prof. J. Scheffer, 212, 213, 214, 215, 216, 217; E. B. Seitz, 212, 213, 214, 215, 216, 217, 219, 220.

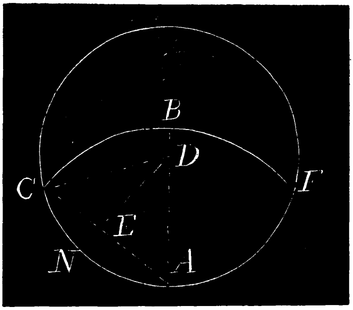
211. " Prove that every number is either a triangular number or is the sum of two, or of three triangular numbers."

[No demonstration of this proposition has been received. Prof. Scheffer writes: "This theorem is a special case of a more general one. As to this case and the following one, viz.: *Every number is either a square or the sum of two, three, or four squares,* the demonstrations have been discovered, but as to the pentagonal and higher numbers, the attempts of the greatest mathematicians at a demonstration have thus far proved futile."

The writer of the article, *number,* in *Johnson's New Encyclopœdia,* says, "It is a general principle, though not capable of rigorous demonstration, that any whole number is equal to the sum of 1, 2 or 3 triangular numbers, or to the sum of 1, 2, 3 or 4 square numbers, or to the sum of 1, 2, 3, 4, or 5 pentagonal numbers, etc."]

212. "One half of a circular tract of land is cut off by an arc of a circle whose center is in the circumference of the circular tract. Find the radius with which the arc is described."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.



Let $AB = x$, $AD = r$ and angle $CAB = \varphi$. The part cut off = sector $ACF + 2$ segment ANC, = sector $ACF +$ 2(sector DAC—triangle DAC); or

$$\tfrac{1}{2}\pi r^2 = x^2\varphi + 2[\tfrac{1}{2}(\pi - 2\varphi)r^2 - \tfrac{1}{2}rx\sin\varphi) \ .(1)$$

But $x = 2r\cos\varphi$. Substituting this value of x in (1) we have

$$\tfrac{1}{2}\pi r^2 = 4r^2\varphi\cos^2\varphi + \pi r^2 - 2r^2\varphi - 2r^2\sin\varphi\cos\varphi,$$

whence

$$\varphi = \frac{r^2\sin 2\varphi - \frac{1}{2}\pi}{4r^2\cos^2\varphi - 2r^2} = \frac{\sin 2\varphi - 1.5708}{2(1 - 2\sin^2\varphi)} . \ (2)$$

From equation (2), by the use of a table of natural sines, the value of φ is found to be 54°35′39″; $\therefore x = 2r\cos\varphi = 1.15872r$.

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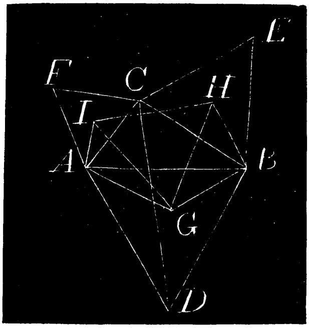
213. "Upon the three sides of any triangle construct equilateral triangles and join their centers by right lines. Prove that the triangle so formed is equilateral."

SOLUTION BY MARCUS BAKER, U. S. COAST SURVEY.

If circles be circumscribed about the three equilateral triangles these three circles will meet in a common point P, such that, A, B and C being the vertices of the triangle, $APB = BPC = CPA = 120°$. Moreover, the lines joining the centers of the circles, two and two, are perpendicular to the common chord of two intersecting circles, viz., PA, PB and PC. Hence these lines by their intersection form an angle of 60°, and the resulting triangle is therefore equiangular and hence equilateral.

Note:—This is apart of Ex. 47 of Chauvenet's Geometry.

[The following demonstration of this proposition is the earliest we have seen, and was made by us and published in the *Ohio Journal of Education* for August, 1857.—Ed.]



Demonstration.—Let ABC represent any triangle, and ADB, BEC and AFC equilateral triangles described on its three sides; and let G, H and I represent, respectively, the centers of the three equilateral triangles.

Join AG, AI, BH, BG, GH, HI, IG, and CD. Then is $\angle GAB = \angle GBA = HBC = \angle IAC = 30°$; and therefore $\angle CBD = \angle HBG$ (each being equal to $\angle HBD - 30°$).

In like manner we may prove that angle $CAD=IAG$. Then, because $BD:BC::BG:BH$, (Eucl. 6, IV) $\angle BGH=\angle BDC$. And in like manner we prove that $\angle AGI=\angle ADC$. Hence $AGI+BGH=60°$. But $AGB=120°$, therefore $HGI=120°—60°=60°$. In precisely the same way we may prove that GHI and HIG each equals $60°$; therefore the triangle GHI is equiangular, and consequently equilateral.

214. "Prove that

$$\cfrac{1}{\tan A+\cfrac{1}{\tan A+\&c.}}=\tfrac{1}{2}[\sqrt{(\sec^2 A+3)}—\tan A]."$$

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

Put $x=\cfrac{1}{\tan A+\cfrac{1}{\tan A+\&c.}}$ Then $x=\dfrac{1}{\tan A+x}$, or $x^2+x\tan A=1$.

Therefore $x=\frac{1}{2}[—\tan A+\sqrt{(\tan^2 A+4)}$. Now $\tan^2 A+4$

$$=\frac{\sin^2 A}{\cos^2 A}+4=\frac{\sin^2 A+\cos^2 A+3\cos^2 A}{\cos^2 A}=\sec^2 A+3.$$

$$\therefore x=\tfrac{1}{2}[\sqrt{(\sec^2 A+3)}—\tan \mathrm{A}].$$

215. "A harbor A is so situated with reference to two headlands B and C, that the angle BAC is a right angle. A ship sails in a course making an angle of 55° with AB, 45 ms. to D, when $DB=DC$: she then sails forward on the same course 15 ms. to E, when BEC is a straight line.

Required AB, AC, DB, EB and EC."

SOLUTION BY PROF. BEMAN, ANN ARBOR, MICH.

Let $AC=x$; $AB=y$; $CD=DB=m$; $AD=a$; $DE=b$; $DAB=\alpha$. From E draw EF perpendicular to AC. By similar triangles,

$$CF:FE::CA:AB, \text{ or } x—(a+b)\sin\alpha:(a+b)\cos\alpha::x:y,$$

$$y=\frac{(a+b)\cos\alpha\, x}{x—(a+b)\sin\alpha}, \qquad (1)$$

$$m^2=x^2+a^2—2a\sin\alpha\, x, \qquad (2)$$

$$m^2=y^2+a^2—2a\cos\alpha\, y. \qquad (3)$$

Subtracting (3) from (2), and substituting values of y as given in (1) we easily obtain

$$x^3 - 2(2a+b)\sin\alpha x^2 + (a+b)[3a - (2a+b)\cos 2\alpha]x - 2a(a+b)^2\sin\alpha = 0.$$

Where $a = 45$, $b = 15$, and $\alpha = 55°$, this becomes

$$x^3 - 210\sin 55° x^2 + 900(9 + 7\sin 20°)x - 324000\sin 55° = 0.$$

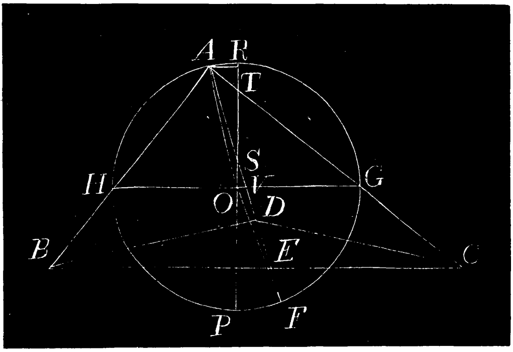
Having found x, y, and m, we can get CE and EB from the relations

$$CE^2 = x^2 + (a+b)^2 - 2(a+b)\sin\alpha x,$$

and $$BE^2 = y^2 + (a+b)^2 - 2(a+b)\cos\alpha y.$$

SOLUTION BY HENRY HEATON, SABULA, IOWA.

The problem may be constructed as accurately as may be desired by the following method.



With O as center, and any radius as OH describe a circle $HAGP$, draw diameters GH and PR at right angles to each other; lay off the arc $HF = 110°$, and on OR lay off $OS_1 = \frac{1}{4}OR$; draw FS_1 and prolong it until it cuts the arc HR in A_1; draw A_1T_1 perpendicular to OR, lay off $OS_2 = \frac{1}{4}OT_1$, draw FS_2 and prolong it until it cuts the arc HR in A_2. Continue this until A_{n-1} is as near A_n as may be desired. Then A_n, T_n and S_n are A, T and S of the figure. On AF lay off $AD = 45$m. and $DE = 15$m.; through E draw BC, parallel to GH, meeting AH and AG prolonged in B and C; then A, B, C, D and E are A, B, C, D and E of the problem.

The demonstration is obvious.

To make the calculation indicated by the construction, we have $OF = r$, $OS_1 = \frac{1}{4}r$, and angle $FOS_1 = 160°$. Angle $POA = 20° =$ the sum and $ROA_1 =$ the difference between the other two angles.

Hence $\tan\frac{1}{2}ROA_1 = \frac{3}{5}\tan 10°$; $\therefore ROA_1 = 12°13'12''$. $OS_2 = \frac{1}{4}OT_1$ $ABC = \frac{1}{4}\cos 12°13'12'' = .241835$.

Proceeding as before we get $\angle ROA_2 = 12°13'26''$, $ROA_3 = 12°13'26''$ &c. Hence angle $ROA = 12°13'26''$. $\therefore AOH = 77°46'34''$; $AGH = 38°53'17''$; $\therefore ABC = 51°06'43''$.

From the triangles AEB, AEC and ADB we find

$AB = 74.0559$ miles,
$AC = 91.8179$ "
$EB = 63.1433$ "
$EC = 54.8177$ " and $BD = 60.7155$ miles.

216. "Through three given points to describe the minimum ellipse."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let A, B, C be the three given points, and let $AB = a$, $AC = b$, angle $BAC = \beta$, and area of triangle $ABC = \Delta$.

Then the equation to the ellipse through A, B, C, referred to AB and AC, is of the form

$$mx^2+ny^2+xy-amx-bny = 0. \tag{1}$$

From (1) we readily find the area of the ellipse to be

$$u = \frac{2\pi mn(a^2m+b^2n-ab)\sin\beta}{\sqrt{(4mn-1)^3}}. \tag{2}$$

Hence we must have

$$\frac{m^2n^2(a^2m+b^2n-ab)^2}{(4mn-1)^3} = \text{a minimum}. \tag{3}$$

Differentiating (3) with respect to m, we have

$$(2mn+1)(a^2m+b^2n-ab) = a^2m(4mn-1). \tag{4}$$

Differentiating (3) with respect to n, we have

$$(2mn+1)(a^2m+b^2n-ab) = b^2n(4mn-1). \tag{5}$$

From (4) and (5) we find $m = \frac{b}{a}$, and $n = \frac{a}{b}$. Substituting the values of m and n in (1), we have

$$b^2x^2+a^2y^2+abxy-ab^2x-a^2by = 0,$$

or $$b^2(x-\tfrac{1}{3}a)^2+a^2(y-\tfrac{1}{3}b)^2+ab(x-\tfrac{1}{3}a)(y-\tfrac{1}{3}b) = \tfrac{1}{3}a^2b^2;$$

hence the centroid of the triangle ABC is the center of the minimum ellipse through A, B, C.

Substituting the values of m and n in (2), we have

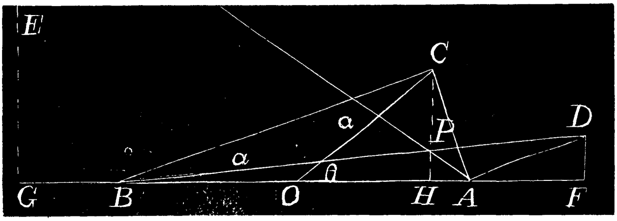
$$u = \tfrac{4}{9}\pi\sqrt{3}.ab\sin\beta = \frac{4\pi}{3\sqrt{3}}\Delta;\ \therefore\ u : \Delta :: 4\pi : 3\sqrt{3}.$$

The maximum inscribed ellipse and the minimum circumscribed ellipse of the triangle are coaxial and similar, the axes of the latter being twice those of the former.

217. "The hypothenuse of a right triangle is fixed, and squares are described upon the other two sides: it is required to find the equation to the locus of the intersection of two straight lines, drawn from either extremity of the hypothenuse to the most distant corner of the square upon the opposite side. Find also the area of the curve."

SOLUTION BY PROF. W. W. JOHNSON, ST. JOHN'S COL., ANNAPOLIS, MD.

Taking the hypothenuse ($=2a$) as axis of x and its middle point as origin, let θ denote the $\angle COA$ in the fig.; the triang's HCA and DAF are equal, and so are the triangles EGB and BHC; hence the coordinates of D are $(a+a\sin\theta,\ a-a\cos\theta)$ and those of E are $(-a-a\sin\theta,\ a+a\cos\theta)$.



Therefore the equation of BD is

$$y=+\frac{1-\cos\theta}{2+\sin\theta}(x+a), \tag{1}$$

and that of AE is

$$y=-\frac{1+\cos\theta}{2+\sin\theta}(x-a). \tag{2}$$

These equations give for the point of intersection P,

$$x=a\cos\theta \text{ and } y=\frac{a\sin^2\theta}{2+\sin\theta}. \tag{3}$$

(The value of x showing that the point P is on CH.) Eliminating θ,

$$y=\frac{a^2-x^2}{2a+\surd(a^2-x^2)},$$

or

$$x^4+x^2y^2+4ax^2y-2a^2x^2+3a^2y^2-4a^3y+a^4=0. \tag{4}$$

The curve is readily traced by means of eq'ns (3). It consists of two branches meeting in cusps at A and B, the tangents at which make angles of 45° with the axis; the lower* branch corresponding to values of θ between 0 and 180°, as in the figure, has a vertex on the axis of y distant $\frac{1}{3}a$ from the origin, the upper branch, corresponding to values of θ between 180° and 360° (the squares being described on the inner sides of the lines AC and BC) has a vertex on the axis of y distant a from the origin.

For the area we have the integral

$$\int y\,dx=-a^2\int\frac{\sin^3\theta\,d\theta}{2+\sin\theta}.$$

Since dx is negative from $\theta=0$ to $\theta=180°$, the integral between these limits will give, negatively, the area between the lower branch and the axis of x; while between the limits 180° and 360° it will give, positively, the area between the upper branch and the axis of x; hence the req'd area is

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*The vertex of the *upper* branch, as represented by the figure, is distant $\frac{1}{3}a$ from the origin, while that of the *lower* branch is distant a.—Ed.

$$-a^2\int_0^{2\pi}\sin^2\theta\,d\theta+2a^2\int_0^{2\pi}\sin\theta\,d\theta-4a^2\int_0^{2\pi}d\theta+8a^2\int_0^{2\pi}\frac{d\theta}{2+\sin\theta}.$$

The value of $\int\frac{d\theta}{2+\sin\theta}$ is $\frac{2}{\sqrt{3}}\tan^{-1}[\sqrt{(\frac{1}{3})}.\tan\frac{1}{2}\theta]$; as the arc $\frac{1}{2}\theta$ passes from 0 to π it is evident that this inverse tangent also passes from 0 to π; hence the area is $(\frac{16}{3}\sqrt{3}-9)\pi a^2$, approximately $.24\pi a^2$.

NOTE, BY THE EDITOR.—Equation (4), in the above solution of problem 217, obviously represents two real and unequal values of y for each value of x, and hence the double sign should be written before the radical; $\therefore$ the area of the upper and lower branches respectively will be represented by

$$\int_{-a}^{a}\frac{(a^2-x^2)dx}{2a+\sqrt{(a^2-x^2)}},\ \text{and}\ \int_{-a}^{a}\frac{(a^2-x^2)dx}{2a-\sqrt{(a^2-x^2)}},$$

and the entire area of the closed curve is

$$\int_{-a}^{a}\left(\frac{a^2-x^2}{2a+\sqrt{(a^2-x^2)}}+\frac{a^2-x^2}{2a-\sqrt{(a^2-x^2)}}\right)dx=\int_{-a}^{a}\frac{4a(a^2-x^2)\,dx}{3a^2+x^2}$$

$$=\tfrac{16}{9}\sqrt{3}.\pi a^2-8a^2,\ \text{approximately}=1.67a^2.$$

The area of the branch which lies above the axis of x is $=\pi a^2(\frac{9}{2}-\frac{16}{9}\sqrt{3})-4a^2$; therefore the area of the bell-shaped branch which lies below the axis of x is $=\pi a^2(\frac{32}{9}\sqrt{3}-\frac{9}{2})-4a^2$; and the area of the hat-shaped curve which lies between the lower curve and the upper curve *inverted*, and represents the *difference* between the two branches, is $=(\frac{16}{3}\sqrt{3}-9)\pi a^2$.

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218. "Required the separate rates of dividend of two insolvent estates connected actually as follows:

John Doe's Estate. Direct liabilities \$33,425.61, also endorsements for Richard Roe \$34,949.16, less primary dividend to be paid from Roe's estate. Net proceeds to be divided \$12,395.76.

Richard Roe's Estate. Direct liabilities \$46,212.00, also endorsements for John Doe \$9,500.00, less primary dividend to be paid from Doe's estate. Net proceeds to be divided \$9,493.25, augmented by dividend on \$4,796.65 account due from John Doe's estate."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Let the rate per cent of dividend on Doe's estate be designated by x, and that on Roe's estate by y.

Put Doe's direct liabilities *plus* his endorsements = \$68374.77 = λ,
Roe's " " " " " = 55712.00 = λ',
Doe's net assets = 12395.76 = a,
Roe's " " = 9493.25 = a',
Roe's account due from Doe's estate = 4796.65 = β.

By the question, $a'+\frac{1}{100}\beta x$ = amount to be divided among Roe's cred's.
$\lambda'-\frac{1}{100}ax$ = Roe's gross liabilities *less* primary dividend from Doe's estate, and

$(\lambda'-\frac{1}{100}ax)\frac{1}{100}y$ = amount which Roe's estate will produce for his creditors, which from the nature of the question must be the the same as $a'+\frac{1}{100}\beta x$.

Equating these two expressions, and reducing,

$$y=\frac{100(100a'+\beta x)}{100\lambda'-ax}.$$

Again $(a'+\frac{1}{100}\beta x)\frac{1}{100}y=\frac{(100a'+\beta x)^2}{100(100\lambda'-ax)}$ = dividend on Roe's estate to be paid to Doe's estate, and

$$\lambda-\frac{(100a'+\beta x)^2}{100(100\lambda'-ax)}$$

= Doe's *gross* liabilities *less* primary dividend from Roe's state, and

$$\left(\lambda-\frac{100a'+\beta x}{100(100\lambda'-ax)}\right)\cdot\frac{x}{100}$$

= dividend on Doe's estate, which, from the nature of the question, must give his net assets = a.

Equating and reducing, there results,

$$x^3+\frac{200a'\beta+100a\lambda}{\beta^2}x^2-\frac{(100)^2\lambda\lambda'+(100)^2a-(100)^2a'^2}{\beta^2}x+\frac{(100)^3a\lambda'}{\beta^2}=0,$$

or, taking the nearest integral coefficients,

$$x^3+4084x^2-1683265x+30015503=0.$$

This equation has three real roots, two positive and one negative; but only one positive root, viz., $x=18.68$ is applicable to the present question. This value of x gives the value of $y=19.46$. Hence the separate rates of dividend are, approximately, 18.68 and 19.46, respectively.

[Mr. Meech finds $x=20.1293$ and $y=19.4399$, but has not given his analysis.]

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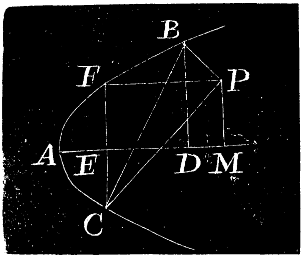
219. "What is the locus generated by the vertex of a right angle, including sides given, sliding between the branches of a given parabola?"

SOLUTION BY E. B. SEITZ.



Let BPC be the right angle, A the vertex of the parabola, AM the axis. Draw PM, BD, CE perpendicular to AM, and PF perpendicular to CE produced.

Let $BP=b$, $CP=c$, $AM=x$, $PM=y$, $\angle BPF=\theta$, and $4a=$ the parameter of the parabola.

Then from the figure we have $BD=y+b\sin\theta$, $AD=x-b\cos\theta$, $CE=y-c\cos\theta$, $AE=x-c\sin\theta$. Hence from the parabola we have

$$(y+b\sin\theta)^2=4a(x-b\cos\theta)^2, \tag{1}$$

$$(y-c\cos\theta)^2=4a(x-c\sin\theta)^2. \tag{2}$$

By eliminating θ between (1) and (2), the equation to the required locus can be found. The equation would apparently be very complicated.

[Mr. Heaton, by eliminating θ from a similar pair of equations, gets

$$\begin{aligned}&(ay-bp)^2\{[ab(by+ap)(a^2bp-2bpx-2abxy-bpy^2-ay^3)+2a^2b^2(p^2+y^2)^2]\\&-2ab(p^2+y^2)\sqrt{[ab(by+ap)(a^2bp-2bpx-2abxy-bpy^2-ay^3)+a^2b^2(p^2+y^2)^2]}\,\}\\&+(by+ap)^2\{[ab(ay-bp)(2ap^2x+2bxy-ab^2p-apy^2-by^3)+2a^2b^2(p^2+y^2)^2]\\&-2ab(p^2+y^2)\sqrt{[ab(ay-bp)(2ap^2x+2bxy-ab^2p-apy^2-by^3)+a^2b^2(p^2+y^2)^2]}\,\}\\&-a^2b^2(by+ap)^2(ay-bp)^2=0.]\end{aligned}$$

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220. "Through a given point in the surface of a circle, a chord is drawn at random. If another chord be drawn at right angles to the first, what is the probability that they will intersect?"

SOLUTION BY PROF. JOHNSON.

Let a be the radius of the given circle, and c the distance of the given point from its centre, and denote by θ the angle between any chord and the diameter passing through the given point; then the length of the chord is

$$2\sqrt{(a^2-c^2\sin^2\theta)}.$$

Assuming that equally distant parallel chords are equally probable, the chance that a line, perpendicular to this chord and known to cut the circle, will cut the chord within the circle, is evidently the ratio of the length of the chord to the diameter. That is, denoting the ratio $c\div a$ by e, the chance is

$$\sqrt{(1-e^2\sin^2\theta}.$$

Hence, assuming all directions of the chord to be equally probable, the required probability is

$$\frac{1}{2\pi}\int_0^{2\pi}\sqrt{(1-e^2\sin^2\theta)},$$

which is the ratio of the perimeter of an ellipse, whose eccentricity is e, to the circle on its major axis ; in other words, the ratio of the perimeter of an inscribed ellipse, with the given point for a focus, to that of the given circle.

[If all *possible* chords are drawn through a point (not the center) within a circle they will not be equally dense around the point; but an indefinite number of chords, or lines, drawn at random *through* a point will be equally dense around the point, hence all values of θ, in the above solution, are, *in fact*, equally probable.

Again, perpendiculars to a chord, equally distant from each other, will not represent all *possible* chords at right angles to that chord; but if in drawing an indefinite number of chords at right angles to the chord first drawn, points be selected on the first chord (which would be likely to be done in practice) through which to draw the second random chords, these chords will be, *in fact*, equally distant from each other.

Mr. Heaton sent two solutions of this problem, corresponding, respectively to the two hypotheses here presented.—Ed.]

PROBLEMS.

221. BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.—Find the four roots of the equation,

$$x^4+px^3+\frac{7p^2}{4^2}x^2+\frac{6p^3}{4^3}x+q=0.$$

222. BY GEORGE NEWBAUER, ANSONIA, OHIO.—It is required to find four positive integral numbers, the sum of the cubes of any three of which shall be a rational cube number.

223. BY WILLIAM HOOVER, BELLEFONTAINE, OHIO.—Compare the masses of two spheres of equal radii, one of uniform density, the other increasing in density from a single point in the surface as the square of the distance measured on the diameter through that point increases.

224. BY GEO. H. HARVILL, BONNER, LA. — Two posts, each 12 feet high, stand perpendicularly, fifty feet apart, on a horizontal plane, and a rope, one end of which is fastened at the top of each post, is suspended between them so that it just touches the plane at its middle point. Required the length of the rope.